COMPLEX ANALYSIS

Sin HITOTUMATU
Kyoto University, JAPAN

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Summary

Complex analysis means the theory of differential and integral calculus for complex valued functions of complex variables. Sometimes "theory of (complex) function" is used in the same meaning. Here "complex" means the complex numbers described below.

1. Complex Numbers

A complex number is an expression of the form $a+ib$ with two real numbers a, b and i is the **imaginary unit**: $i=\sqrt{-1}$.

The set of all complex numbers C forms a field, i.e. an algebraic system admitting four arithmetic operations as usual, except division by 0. The arithmetic operations for complex numbers may be performed as a polynomial of i, provided that i^2 is replaced by –1.

For example, division can be carried out by the formula

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}.$$

The complex number $a+ib$ is represented by a point with the coordinates (a, b) over a plane. The plane on which complex numbers are represented is called the **complex plane** or the **Gauss-Argand plane (**See Figure 1)

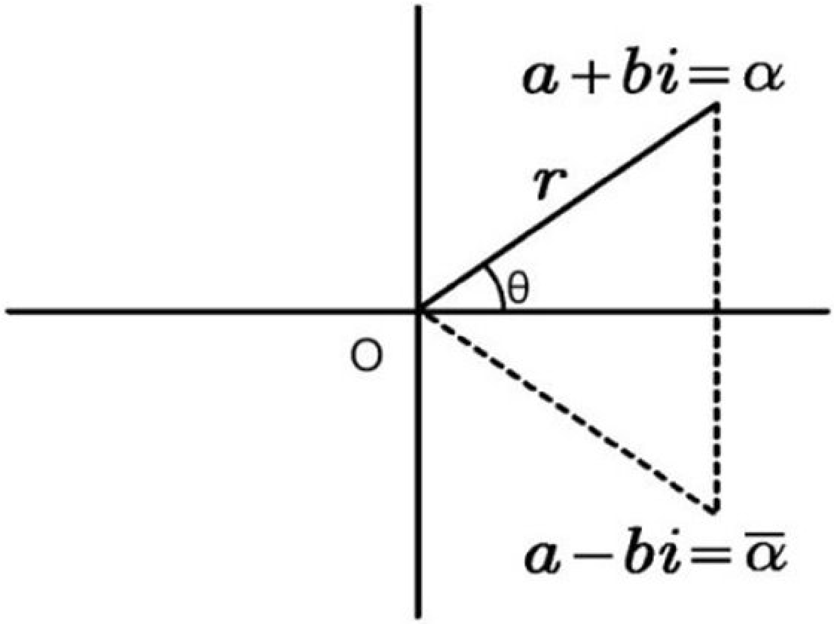

Figure 1: Gauss-Argand plane

A complex variable is usually denoted by $z = x + iy$, where x and y are real variables, called the **real part** and the **imaginary part** of z respectively. When we take the polar coordinates on the complex plane the magnitude or the radius is denoted by

$$|a| = |a+ib| = \sqrt{a^2+b^2}$$

and is called the **absolute value** of $\alpha = a + ib$. The **argument** is denoted by arg α. A complex number α with $|\alpha| = 1$ is expressed as

$$\alpha = \cos\theta + i\sin\theta, \quad \theta = \arg\alpha.$$

The above expression is usually written as $\exp(i\theta)$ or $e^{i\theta}$. Sometimes an abbreviated notation $\cos(\theta)$ is used. According to the addition theorem we have

$\exp(i(\theta+\phi)) = \exp(i\theta)\cdot\exp(i\phi)$, $(\exp(i\theta))^n = \exp(in\theta)$. These relations are called **de Moivre theorem**.

In some cases, we use the following compactification. Let Σ be a sphere in (ξ, η, ζ)-space of radius 1 with the center at the origin. Let its equatorial plane $\zeta = 0$

be a complex plane C, whose real and the imaginary axes are the ξ and η axes respectively. A complex number $z = x + iy$ is represented by a point Z on C. A straight line from the north pole N(0, 0, 1) to Z intersect Σ at another point $P(\xi,\eta,\zeta)$. The mapping Z→P is called a **stereographic projection** and is given by the following relations.

$$z = \frac{\xi + i\eta}{1-\zeta}; \quad \xi = \frac{2x}{1+|z|^2}, \eta = \frac{2y}{1+|z|^2}, \quad \zeta = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

There is no point on C corresponding to the north pole N itself, but we add a new element denoted by ∞, called the **point at infinity**, which is assumed to be the image of N. Then Σ corresponds to $\bar{\mathrm{C}} = \mathrm{C} \cup \{\infty\}$ and is called the **complex sphere** or **Riemann sphere** (see Figure 2). When we discuss a function near ∞, we take usually the local variable $w = 1/z$ around ∞.

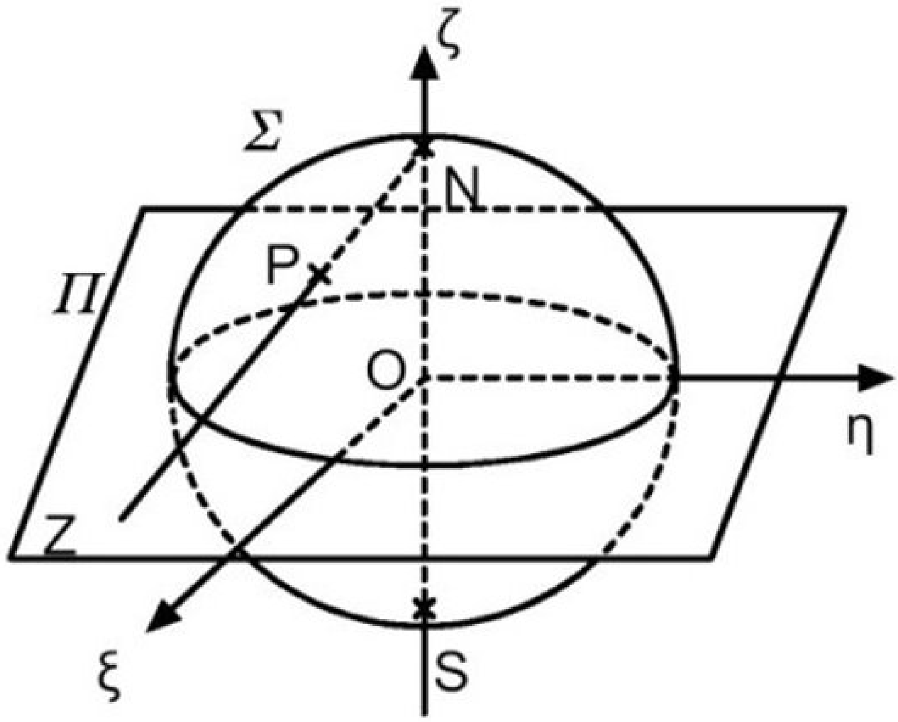

Figure 2: Complex sphere or Riemann sphere

2. Holomorphic Functions

Let $f(z)$ be a complex valued function defined in an open set D on the complex plane C. $f(z)$ is called **differentiable** at z, if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

exists and is finite as the complex increment h tends to 0. $f'(z)$ is called the **derivative** of $f(z)$ in D. Though the formal definition is similar to a function of a real variable, it is much stronger than the case of real functions, since $z+h$ may be an arbitrary point in a 2-dimensional neighborhood of z. In particular, if $f'(z) \neq 0$, we have

$$\frac{f(z+h)-f(z)}{f(z+k)-f(z)} : \frac{h}{k}$$

for two neighboring points $z+h$ and $z+k$ near z. This relation implies that the image under f of the triangle with vertices at z, $z+h$ and $z+k$ is approximately similar to the original one, i.e., f is **conformal** or angle-preserving mapping.

A function $f(z)$ differentiable at each point of an open set *D* is called **holomorphic** or **regular** in *D*.

Almost all orthogonal curvilinear coordinates on a plane are given from the Cartesian or the polar coordinates on the plane by conformal mappings of various holomorphic functions. For example, the function $w = z^2$ maps the lines $x = a$ and $y = b$ $\left(z = x + iy;\, a, b\right)$; being real constants) to a parabola open to the right and to the left, respectively. The symmetric axis of the parabola is the real axis and the focus lies at the origin (see Figure 3). This gives the **parabolic coordinates**.

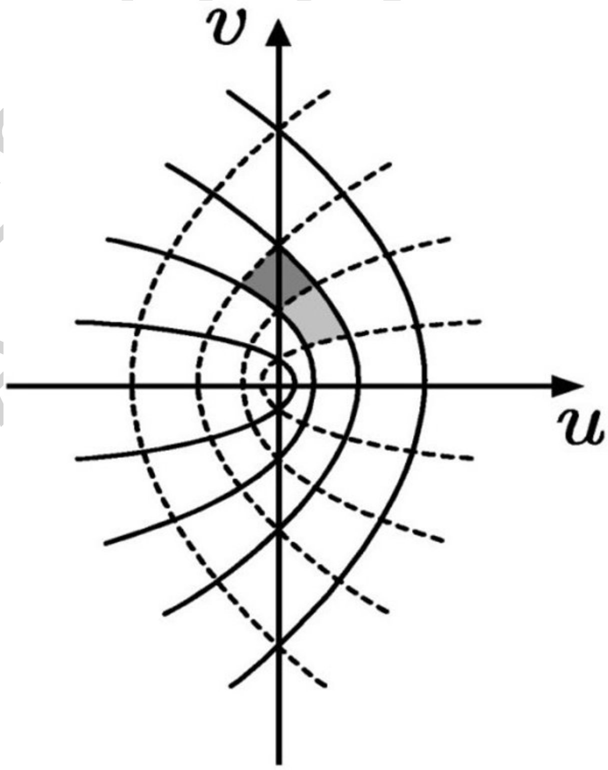

Figure 3: Parabolic coordinates

Another example is the function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$. This maps a circle $|z| = r$ and a ray $\arg z = \theta$ to an ellipse and a hyperbola, respectively. Their foci are at +1 and –1.

This gives rise to the **elliptic coordinates** (see Figure 4).

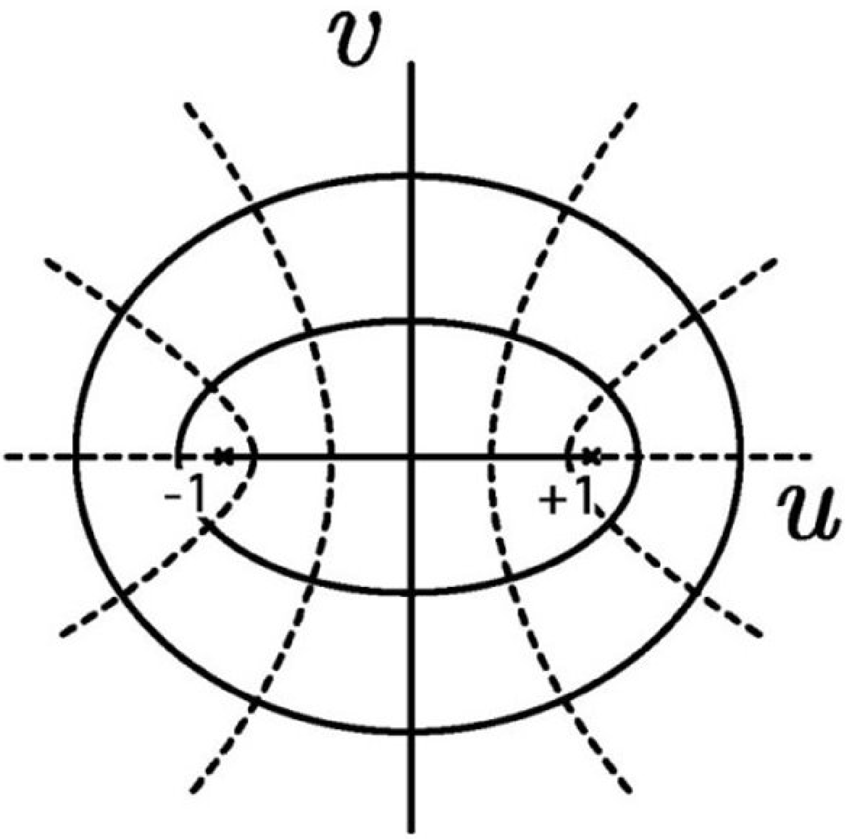

Figure 4: Elliptic coordinates

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Biographical Sketch

Professor Sin HITOTUMATU was born on March 6, 1926. He received BS and PhD degrees from University of Tokyo in 1947 and 1954 respectively. From August 1957 to September 1958 he visited Stanford University under Fulbright Grant. Professor HITOTUMATU's professional career is characterized by the following positions:

1952 June—1955 October: Associate Professor, St. Paul (Rikkyo) Univ., Tokyo
1955 November—1962 March: Associate Professor, Univ. of Tokyo
1962 April—1969 March: Professor, St. Paul Univ. Tokyo
1969 April—1989 March: Professor, Research Institute for Mathematical Sciences, Kyoto University.
1989 March: Retired , Professor Emeritus, Kyoto Univ.
1989 April—1996 March: Professor, Tokyo Denki Univ., Hatoyama Campus,
1996 April—2002 March: Visiting Professor, Tokyo Denki Univ., Hatoyama Campus
1995 April—2002 July: President of The Mathematics Certification Association (SUKEN) of Japan